Observer-based Fault Estimation and Tolerant Control for Stochastic Takagi-Sugeno Fuzzy Systems with Brownian Parameter Perturbations ⋆

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Abstract

In this paper, robust fault estimation and fault tolerant control for stochastic Takagi-Sugeno fuzzy systems, subjected to Brownian parameter perturbations, unknown process uncertainties and unexpected faults, are investigated. Augmented system approach, unknown input observer techniques and sliding mode control strategies are integrated to decouple the influences from the unknown input uncertainties, and drive the trajectories of the estimation error dynamics to enter and subsequently remain within a desired surface of the error space. As a result, a robustly simultaneous estimate of the means of the faults concerned and the full system states can be achieved. In the meanwhile, the actuator/sensor signal compensation techniques are used to formulate the tolerant control strategy to eliminate or offset the influences from the faults to the systems dynamics and ensure the robust stabilization of the closed-loop control system. In terms of linear matrix inequalities, sufficient conditions are proposed to ensure the robust stability of the overall closed-loop system composed of system state and estimation error dynamics, as well as the reachability of the sliding mode surface. Furthermore, the systematic design procedures for the robust fault estimation and fault tolerant control scheme are addressed. Finally, simulation studies on a single-link manipulator and a three-tank system are illustrated to demonstrate the effectiveness of the suggested methodologies.

Key words: Fault estimation;fault tolerant control;stochastic Takagi-Sugeno systems;unknown input observer;sliding mode control.

1 Introduction

In response to high-demand requirements on system availability and safety in industrial automation systems, fault diagnosis and tolerant control has long been a hot issue worldwide, and comprehensive research results were documented during the past decades (e.g., see [26], [5], [28], [8], and the references therein). Fault tolerant control (FTC) aims to preserve the functionalities of a faulty system with acceptable performance. With the aid of an advanced fault diagnosis block providing the online information of faulty features, active FTC can real-time reconfigure controller laws such that the influences from the unforeseen faults to the system dynamics are eliminated/relieved, and the system stability is guaranteed. Fault estimation can provide rich knowledge about faults and generate auxiliary full state estimation as a by-product. In consequence, it is motivated to develop integrated fault estimation and fault tolerant control techniques for industrial dynamics to make the system resilient to unexpected faults. During the past decades, a variety of approaches such as sliding mode observer based method (e.g., see [27]), adaptive observer-based method ([6]), and descriptor observer based method (e.g. [9]), were developed for fault estimation and tolerant control.

Owing to the widespread presence of random factors in the operation of systems, stochastic systems formulated

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in Itô-type stochastic differential equations ([21], [11]) have played crucial roles in modelling practical systems such as nuclear systems, chemical processes, biology systems and thermal systems. On the other hand, nonlinear systems have attracted much attention in the control community as they have a better accuracy for describing industrial processes. In the meanwhile, it has brought challenges in analysis and control for a complex system described by a high-nonlinear mathematical model. Takagi-Sugeno (T-S) fuzzy models (see [24]), capable of providing the approximation of nonlinear characteristics by fuzzy blending of several local linear models with appropriate membership functions, were thus motivated and have become a powerful tool to simplify the analysis and control for nonlinear processes (e.g., see [22]). In order to resolve the analysis and control synthesis issue for a complex industrial process subjected to stochastic phenomena and high nonlinearities, a natural idea is to develop stochastic T-S fuzzy model based approach and techniques. Recently, there were some interesting results reported on fault diagnosis and fault tolerant control based on T-S fuzzy models with Brownian motions (e.g., see [18]).

In real engineering systems, process disturbances and uncertainties are unavoidable. The robustness is thus a crucial issue in the field of fault diagnosis and tolerant control. In order to decouple the process uncertainties, one of the popular techniques is the unknown input observer (UIO) which has a widespread application in the field of robust fault diagnosis (e.g., see [16], [13]). The conventional unknown input observer schemes assume the process uncertainties can be decomposed completely. Unfortunately, this assumption is not always true in practical engineering systems. In the paper by [10], a novel UIO was proposed for fault estimation for a class of deterministic nonlinear systems, and the results were extended to a class of nonlinear systems with Brownian motions (see [19]). Another powerful technique for disturbance attenuation is sliding mode control (SMC) where a discontinuous term is used to enable the observer to reject unknown input effects (see [25], [1], and [2]). Therefore, it is of interest to integrate the UIO and SMC methods to enhance the perturbations attenuation ability. To the best of our knowledge, few effort has been devoted to robust sliding mode unknown input observer based fault estimation and fault tolerant control for stochastic nonlinear systems, particularly for nonlinear systems subjected to Brownian parameter perturbations, which remains to be an area to be explored. In this paper, an innovative fault estimation and fault tolerant control approach is developed by integrating augmented system approach, unknown input observe, sliding mode control and linear matrix inequality optimization for stochastic T-S fuzzy systems.

The rest of paper is organized as follows. Problem formulation and some preliminary knowledge are given in Section 2. Robust fault estimation approach of stochastic T-S fuzzy systems is investigated in Section 3. Robust fault tolerant control strategy is addressed in Section 4, along with the discussion of reachability of the sliding mode surface and a summary of design procedures of the suggested methods. Extension to stochastic fuzzy system with uncertainties and faults in stochastic perturbation can be found in section 5. Simulation validations on a single-link manipulator system and a three-tank model are developed in Section 6. The paper is ended by the conclusion and future work in Section 7.

2 Preliminaries and problem formulation

In this paper, the following notations are used. $\mathcal{R}^n$ and $\mathcal{R}^{n\times m}$ stand for n-dimentional Euclidean space and $n\times m$ real matrices, while $\mathcal{R}^+$ is the set of all nonnegative real numbers.I_n represents identity matrix with dimension of $n\times n$, and 0 denotes a scalar zero or a zero matrix with appropriate zero entries. For any given vector $x=(x_1,x_2,...x_n)\in\mathcal{R}^n$, $\|x\|$ refers to its Euclidean norm defined by $\|x\|=(\Xi_{i=1}^n x_i^2)^{1/2}$ and $\|x\|_{Tf}=(\int_0^{Tf}x^T(\tau)x(\tau)d\tau)^{1/2}$, while $|\cdot|$ denotes 1-norm of a vector. $\mathcal{E}(x)$ represents mathematical expectation of x and $\forall$ means for all. The superscript T represents the transpose of matrices or vectors. The notation $X>0$ indicates that the symmetric X is positive definite. Besides, in a large matrix expression, $\begin{bmatrix} G_1 & G_2 \\ * & G_3\end{bmatrix}=\begin{bmatrix} G_1 & G_2 \\ G_2^T & G_3\end{bmatrix}$.

Consider stochastic T-S fuzzy models suffering from faults and unknown inputs in the form of Itô -type differential equations as follows:

IF μ_1 is M_{1i} and ...$\mu_q$is M_{qi}, THEN

$$\begin{cases} dx(t)=[A_ix(t)+B_iu(t)+B_{di}d(t) \\ \qquad +B_{fi}f(t)]dt+W_ix(t)d\omega(t) \\ y(t)=Cx(t)+D_ff(t)\end{cases} \tag{1}$$

where $x(t)\in\mathcal{R}^n$ represents unmeasurable state vector with measurable initial value of $x_0\in\mathcal{R}^n$ at initial time t_0; $\omega(t)$ is a standard one-dimensional Brownian motion on the complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq t_0},\mathcal{P})$, with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\mathcal{F}$ being a filtration and $\mathcal{P}$ being a probability measure. $\omega(t)$ satisfies $\mathcal{E}[\omega(t)]=0$ and $\mathcal{E}[\omega^2(t)]=t$. $u(t)\in\mathcal{R}^m$ stands for control input vector and $y(t)\in\mathcal{R}^p$ is measurement output vector; $d(t)\in\mathcal{R}^{l_d}$ is the bounded deterministic unknown input vectors; $f(t)\in\mathcal{R}^{l_f}$ represents the means of faults (actuator and/or sensor faults).When $f(t)=f_a(t)$, where $f_a(t)$ is actuator fault, then $B_{fi}=B_i,D_f=0^{p\times l_f}$;when $f(t)=f_s(t)$, where $f_s(t)$ is sensor fault, then $B_{fi}=0^{n\times l_f},D_f=I_p$; when $f(t)=\left[\, f_a^T(t)\;\; f_s^T(t)\,\right]^T$,then $B_{fi}=\left[\, B_{fai}\;\; B_{fsi}\,\right],D_f=$

$[\, D_{fai} \ D_{fsi} \,]$, where B_{fai} and D_{fai} represent the coefficients of actuator fault, B_{fsi} and D_{fsi} are coefficients of sensor fault. $i = 1, 2, ...r$, and r is the total number of local models, M_{ji} are fuzzy sets and decision vector μ involves all individual premise variables $\mu_j, j = 1, 2, ...q$, which can be measured. $A_i, B_i, C, B_{di}, B_{fi}, W_i, D_f$ are known coefficient matrices with appropriate dimensions. In the rest of paper, the symbol t in equations will be omitted for the simplicity of presentation.

By using the standard fuzzy blending method, the global model of system (1) can be inferred as:

$$\begin{cases} dx = \Xi_{i=1}^{r} h_i(\mu)[(A_i x + B_i u + B_{di} d + B_{fi} f)dt \\ \qquad + W_i x d\omega] \\ y = Cx + D_f f \end{cases} \tag{2}$$

where $h_i(\mu)$ are membership functions, following the convex sum properties: $\Xi_{i=1}^{r} h_i(\mu) = 1$ and $0 \leq h_i(\mu) \leq 1$.

Definition 1 ([11]). For stochastic system (2), the solution is stochastic exponentially stable in mean square, if there exist positive scalars α and β such that

$$\mathcal{E}(\|x(t)\|^2) \leq \alpha e^{-\beta t} \tag{3}$$

Definition 2 ([23]). For stochastic system

$$dx = l(t, x)dt + h(t, x)d\omega \tag{4}$$

where $l(t, x)$ and $h(t, x)$ stand for system dynamic function and stochastic perturbation distribution function, respectively, given any function $V(t, x) \in C^{2\times 1}\{\mathcal{R}^n \times [t_0, \infty] \to \mathcal{R}^+\}$, the infinitesimal generator $\mathcal{L}V(t, x)$ is defined as:

$$\begin{aligned} \mathcal{L}V(t, x) = &\frac{\partial V(t, x)}{\partial t} + \left[\frac{\partial V(t, x)}{\partial x}\right]^T l \\ &+ \frac{1}{2} trace\{h^T \frac{\partial^2 V(t, x)}{\partial x^2} h\} \end{aligned} \tag{5}$$

Lemma 1 ([21]). If there exist a function V, and positive constants c_1, c_2 and c_3, such that

$$c_1 \mathcal{E}(\|x\|^2) \leq \mathcal{E}(V) \leq c_2 \mathcal{E}(\|x\|^2) \tag{6}$$

and

$$\mathcal{E}(\mathcal{L}V(t, x)) \leq c_3 \mathcal{E}(\|x\|^2) \tag{7}$$

then the solution of stochastic system (2) is stochastically exponentially stable in mean square.

Lemma 2 ([4], Schur complement). Let $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$ be a symmetric matrix. $S < 0$ is equivalent to $S_{22} < 0$ and $S_{11} - S_{12} S_{22}^T S_{12}^T < 0$.

The main objective of this paper is to design robust fault estimation and fault tolerant control strategies for stochastic system (2). Augmented system approach, unknown input observer technique, and sliding mode control method, are integrated to generate robustly simultaneous estimates of the system states and faults. Based on the estimated states and faults, fault tolerant control is implemented by using sensor signal compensation, actuator signal compensation, and robust control. Fig. 1 depicts the scheme of the proposed fault estimation and tolerant control.

3 Joint robust state/fault estimation scheme for stochastic Brownian systems

In this section, the methodology to design a robust fault estimation approach for system (2) is presented. In order to estimate the means of faults and system states at the same time, an auxiliary system is constructed as follows, by considering the faults as augmented system states:

$$\begin{cases} d\bar{x} = \Xi_{i=1}^{r} h_i(\mu)[(\bar{A}_i \bar{x} + \bar{B}_i u + \bar{B}_{di} d + J_1 \ddot{f})dt + \bar{W}_i x d\omega] \\ y = \bar{C}\bar{x} \end{cases} \tag{8}$$

where

$$\begin{aligned} &\bar{n} = n + 2l_f, \bar{x} = [x^T \ \ \dot{f}^T \ \ f^T]^T \in \mathcal{R}^{\bar{n}}, \\ &\bar{A}_i = \begin{bmatrix} A_i & 0_{n\times l_f} & B_{fi} \\ 0_{l_f\times n} & 0_{l_f\times l_f} & 0_{l_f\times l_f} \\ 0_{l_f\times n} & I_{lf} & 0_{l_f\times l_f} \end{bmatrix} \in \mathcal{R}^{\bar{n}\times\bar{n}}, \\ &\bar{B}_i = [B_i^T \ \ 0_{m\times l_f} \ \ 0_{m\times l_f}]^T \in \mathcal{R}^{\bar{n}\times m}, \\ &\bar{B}_{di} = [B_{di}^T \ \ 0_{l_d\times l_f} \ \ 0_{l_d\times l_f}]^T \in \mathcal{R}^{\bar{n}\times l_d}, \\ &\bar{C} = [C \ \ 0_{p\times l_f} \ \ D_f] \in \mathcal{R}^{p\times\bar{n}}, \\ &\bar{W}_i = [W_i^T \ \ 0_{n\times l_f} \ \ 0_{n\times l_f}]^T \in \mathcal{R}^{\bar{n}\times n}, \\ &J_1 = [0_{l_f\times n} \ \ I_{l_f\times l_f} \ \ 0_{l_f\times l_f}]^T \in \mathcal{R}^{\bar{n}\times l_f}. \end{aligned}$$

In system (8), the components of state vector $\bar{x}$ include original state x, the means of concerned faults, denoted by f, and the first-order differentials of the means of the faults, that is $\dot{f}$. In this paper, we assume that $\ddot{f}$ is not zero but bounded and satisfies:

$$\|\ddot{f}\| \leq \delta \tag{9}$$

where δ is a positive scalar. Condition (9) means the variation speed of fault is bounded by δ. In this work, we consider δ is not very big, which means the faults do not

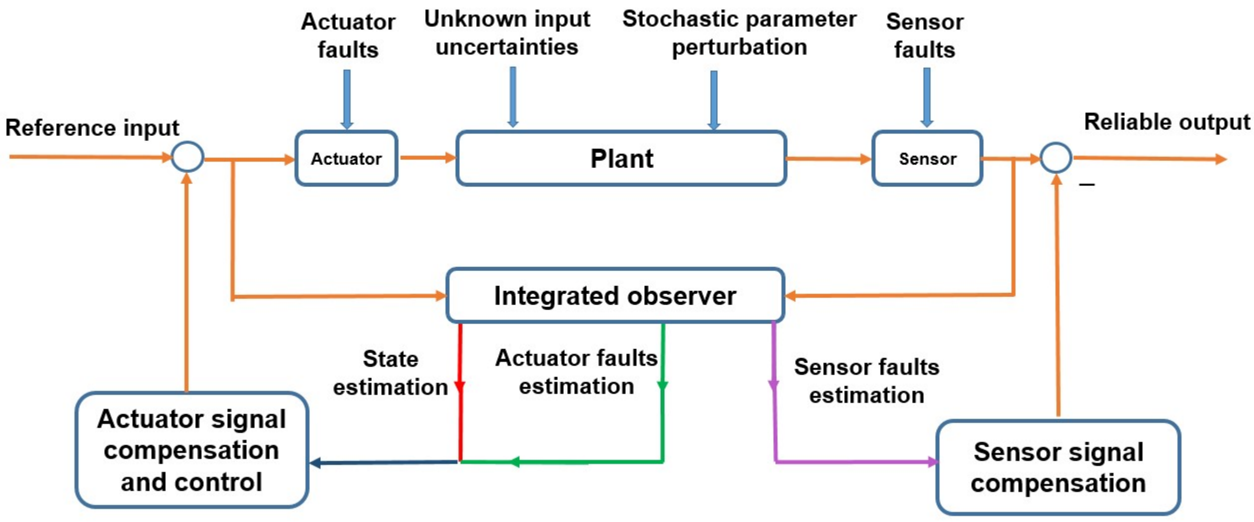

Fig. 1. The structure of observer-based fault tolerant control

vary in high speed. This meets most operation situation of practical systems. In addition, we suppose that d can be decoupled, whereas $\ddot{f}$ cannot.

For system (8), the following fuzzy unknown input observer consisting sliding mode terms is constructed:

$$\begin{cases} d\bar{z} = \Xi_{i=1}^{r} h_i(\mu)[R_i\bar{z} + TB_iu + (L_{i1} + L_{i2})y + L_{si}u_s]dt \\ \hat{\bar{x}} = \bar{z} + Hy \end{cases} \tag{10}$$

where $\bar{z}$ is a state vector of the observer (10), $\hat{\bar{x}}$ is the estimation of $\bar{x}$, u_s is a discontinuous control input to compensate the influences of $\ddot{f}$. The matrices $H, R_i, T, L_{i1}, L_{i2}$ and L_{si}, where $i = 1, 2, ...r$, are observer gains to be designed such that $\hat{\bar{x}}$ is close enough to $\bar{x}$.

Remark 1. Using nonlinear discontinuous term u_s, the designed sliding mode unknown input observer constraints the trajectory of the estimation error to remain on a specific surface such that error is insensitive to the disturbances.

Defining estimation error as $\bar{e} = \bar{x} - \hat{\bar{x}}$, and subtracting (10) from (8) leads to the following error dynamic:

$$\begin{aligned} d\bar{e} = \Xi_{i=1}^{r} h_i(\mu)\{&\{(\bar{A}_i - H\bar{C}\bar{A}_i - L_{i1}\bar{C})\bar{e} \\ &+ (\bar{A}_i - H\bar{C}\bar{A}_i - L_{i1}\bar{C} - R_i)\bar{z} \\ &+ [(\bar{A}_i - H\bar{C}\bar{A}_i - L_{i1}\bar{C})H - L_{i2}]y \\ &+ [(I_{\bar{n}} - H\bar{C}) - T]\bar{B}_iu + (I_{\bar{n}} - H\bar{C})\bar{B}_{di}d \\ &+ (I_{\bar{n}} - H\bar{C})J_1\ddot{f} - L_{si}u_s\}dt \\ &+ (I_{\bar{n}} - H\bar{C})\bar{W}_ix d\omega\} \end{aligned} \tag{11}$$

If the observer gains satisfy the following conditions:

$$(I_{\bar{n}} - H\bar{C})\bar{B}_{di} = 0 \tag{12}$$

$$R_i = \bar{A}_i - H\bar{C}\bar{A}_i - L_{i1}\bar{C} \tag{13}$$

$$T = I_{\bar{n}} - H\bar{C} \tag{14}$$

$$L_{i2} = R_iH \tag{15}$$

the state estimation error can be reduced to

$$d\bar{e} = \Xi_{i=1}^{r} h_i(\mu)[(R_i\bar{e} + TJ_1\ddot{f} - L_{si}u_s)dt + T\bar{W}_ix d\omega] \tag{16}$$

In order to meet the conditions (12) to (15), we have the following assumptions in terms of the references ([5]; [10]):

(1) $rank(\bar{C}(\bar{B}_{d1} \;\; \bar{B}_{d2} ... \; \bar{B}_{dr})) = rank(\bar{B}_{d1} \;\; \bar{B}_{d2} ... \; \bar{B}_{dr})$;

(2) $\forall i, \begin{bmatrix} A_i & B_{fi} & B_{di} \\ C & D_f & 0 \end{bmatrix}$ is of full column rank;

(3) $\forall i, rank \begin{bmatrix} sI_n - A_i & B_{di} \\ C & 0 \end{bmatrix} = n + l_d.$

The above assumptions are to ensure that for each local model, equation (12) can be solved, and one solution of H can be obtained as

$$H^* = \bar{B}_U[(\bar{C}\bar{B}_U)^T(\bar{C}\bar{B}_U)]^{-1}(\bar{C}\bar{B}_U)^T \tag{17}$$

where $\bar{B}_U$ is of full column rank, obtained by a non-singular matrix U such that

$$(\bar{B}_{d1} \;\; \bar{B}_{d2} ... \; \bar{B}_{dr})U = (\bar{B}_U \;\; 0) \tag{18}$$

Moreover, the model is observable.

Under these necessary assumptions, d has been decoupled by selecting suitable H. However, $\ddot{f}$ still affect the error dynamic. As a good observer should lead to the convergence of error $\bar{e}$, the design of robust fault estimation scheme is converted into attenuating the influences of $\ddot{f}$. Now it is time to discuss how to design the sliding mode controller u_s to eliminate its influence.

$\forall i$, the coefficient of sliding mode controller is chosen to be $L_{si} = TJ_1$. Define a sliding mode surface

$$s = J_1^T T^T \bar{P}\bar{e} = 0 \tag{19}$$

where $\bar{P}$ is a positive matrix satisfying

$$J_1^T T^T \bar{P} = N\bar{C} \tag{20}$$

while N is a parameter matrix to be designed. The sliding surface is defined according to the desired dynamical specifications of the closed-loop error dynamic. Here, the estimation error can be driven around zero on surface (19). To make sliding mode achieve and maintain on the surface, the discontinuous input law is introduced as follows

$$u_s = (\delta + \varepsilon)sgn(s) \tag{21}$$

where ε is a parameter to be designed later.

Remark 2. Since $\bar{e}$ is unknown, s is not available from (19). Constraint (20) is introduced such that $s = N\bar{C}\bar{e} = N\bar{C}\bar{x} - N\bar{C}\hat{\bar{x}} = Ny - N\bar{C}\hat{\bar{x}}$, where y and $\bar{C}\hat{\bar{x}}$ are available in practice.

It is noted that the estimation error dynamics are also subjected to stochastic parameter perturbations. Therefore, it is infeasible to design the observer gain independent of the control gains. The integrated design of the observer gain and control gain will be addressed in the next section.

4 Robust fault tolerant control scheme

4.1 Design of observer-based control law

Now we consider the design of the control law for system (2), to compensate the effects of the faults, make the trajectories of state stable, and eliminate the influences of unknown inputs.

Firstly, we introduce the following feedback gain $[K_i \ \ 0 \ \ K_f]$ and construct u in the following fuzzy form:

$$\begin{aligned} u &= \Xi_{i=1}^r h_i(\mu)\bar{K}_i\hat{\bar{x}} \\ &= \Xi_{i=1}^r h_i(\mu)[K_i \ \ 0 \ \ K_f] \begin{bmatrix} \hat{x} \\ \hat{\dot{f}} \\ \hat{f} \end{bmatrix} \\ &= \Xi_{i=1}^r h_i(\mu)(K_i\hat{x} + K_f\hat{f}) \end{aligned} \tag{22}$$

where

$$K_f = -B_h^+ B_{fh} \tag{23}$$

with$B_h = \Xi_{i=1}^r h_i(\mu)B_i, B_{fh} = \Xi_{i=1}^r h_i(\mu)B_{fi}$, and supposing $rank[B_h \ \ B_{fh}] = rankB_h$.

Then it is clear that

$$\Xi_{i=1}^r h_i(\mu)(B_{fi} + B_iK_f) = B_{fh} + B_hK_f = 0 \tag{24}$$

In order to eliminate the effects caused by sensor faults, the following sensor compensation output is employed:

$$y_c = y - D_fJ_2\hat{\bar{x}} = Cx + D_ff - D_f\hat{f} = Cx + D_fJ_2\bar{e} \tag{25}$$

where $J_2 = [0_{l_f\times n} \ \ 0_{l_f\times l_f} \ \ I_{l_f}]$, and y_c is called reliable output.

Substituting (22) into systems (2) and using the reliable output y_c to replace the actual measurement y, the following closed-loop system can be formulated

$$\begin{cases} dx = \Xi_{i=1}^r h_i(\mu)\Xi_{j=1}^r h_j(\mu)[((A_i + B_iK_j)x - B_{eij}\bar{e} \\ \qquad + B_{di}d)dt + W_ixd\omega] \\ y_c = Cx + D_e\bar{e} \end{cases} \tag{26}$$

where$B_{eij} = B_iK_jJ_0 - B_{fi}J_2, J_0 = [I_n \ \ 0_{n\times l_f} \ \ 0_{n\times l_f}]$ and $D_e = D_fJ_2$. It can be seen that the fault estimation performance will have an impact on the fault tolerant control performance.

Under the designed observer-based fault tolerant controller, the overall closed-loop system can be obtained as follows:

$$\begin{cases} dx = \Xi_{i=1}^r h_i(\mu)\Xi_{j=1}^r h_j(\mu)[((A_i + B_iK_j)x - B_{eij}\bar{e} \\ \qquad + B_{di}d)dt + W_ixd\omega] \\ d\bar{e} = \Xi_{i=1}^r h_i(\mu)[(R_i\bar{e} + TJ_1\ddot{f} - L_{si}u_s)dt + T\bar{W}_ixd\omega] \\ y_c = Cx + D_e\bar{e} \end{cases} \tag{27}$$

The next step is to choose appropriate observer and controller gains to make the closed loop system stable, and satisfy the following performance:

$$\mathcal{E}(\|y_c\|_{Tf}^2) < \gamma^2\mathcal{E}(\|d\|_{Tf}^2) \tag{28}$$

where $\gamma > 0$ is the H_∞ performance index.

Then we have the following theorem.

Theorem 1. For system (2), there exists a tolerant observer-based controller in the form of (10), (22) and (25), making closed-loop system (27) be stochastic exponentially stable in mean square and satisfy $\mathcal{E}(\|y_c\|^2_{Tf}) < \gamma^2\mathcal{E}(\|d\|^2_{Tf})$, if $\forall i, j$, there exist positive definite matrices P and $\bar{P}$, matrices K_j and L_{i1} such that

$$\begin{bmatrix} \Omega_{ij11} & PB_{di} & -PB_{eij} + C^T D_e \\ * & -\gamma^2 I_{ld} & 0 \\ * & * & \Omega_{ij33} \end{bmatrix} < 0 \tag{29}$$

and

$$J_1^T T^T \bar{P} = N\bar{C} \tag{30}$$

where $\Omega_{ij11} = P(A_i + B_iK_j) + (A_i + B_iK_j)^T P + W_i^T PW_i + \bar{W}_i^T T^T \bar{P}T\bar{W}_i + C^T C, \Omega_{ij33} = \bar{P}T\bar{A}_i + \bar{A}_i^T T^T \bar{P} - \bar{P}L_{i1}\bar{C} - \bar{C}^T L_{i1}^T \bar{P} + D_e^T D_e$, and $i, j = 1, 2, ...r$.

Proof. Based on Lemma 1, the proof involves establishing a dissipation inequality via a suitable storage function. Here, we choose the function as $V = V_1 + V_2$, where $V_1 = x^T Px$ and $V_2 = \bar{e}^T P\bar{e}$. We can notice

$$V = \tilde{x}^T \tilde{P}\tilde{x} \tag{31}$$

where $\tilde{x} = [x^T \;\; \bar{e}^T]^T, \tilde{P} = \begin{bmatrix} P & 0 \\ 0 & \bar{P} \end{bmatrix}$. It is not hard to find that

$$\lambda_{min}(\tilde{P})\mathcal{E}(\|\tilde{x}\|^2) \le \mathcal{E}(V) \le \lambda_{max}(\tilde{P})\mathcal{E}(\|\tilde{x}\|^2) \tag{32}$$

which implies V satisfy (6) in Lemma 1. Taking infinitesimal generator along the state trajectories of (27), by using It$\hat{o}$ formula, it follows that:

$$\begin{aligned} \mathcal{L}V_1 = &\Xi^r_{i=1}h_i(\mu)\Xi^r_{j=1}h_j(\mu)\{x^T[P(A_i + B_iK_j) \\ &+ (A_i + B_iK_j)^T P]x - 2x^T PB_{eij}\bar{e} \\ &+ 2x^T PB_{di}d + x^T W_i^T PW_ix\} \end{aligned} \tag{33}$$

and

$$\begin{aligned} \mathcal{L}V_2 = &\Xi^r_{i=1}h_i(\mu)\{[\bar{e}^T(\bar{P}R_i + R_i^T \bar{P})\bar{e} - 2\bar{e}^T \bar{P}L_{si}u_s \\ &+ 2\bar{e}^T \bar{P}TJ_1\ddot{f}] + x^T \bar{W}_i^T T^T \bar{P}T\bar{W}_ix\} \end{aligned} \tag{34}$$

Then we have

$$\begin{aligned} \mathcal{L}V &= \mathcal{L}V_1 + \mathcal{L}V_2 \\ &= \Xi^r_{i=1}h_i(\mu)\Xi^r_{j=1}h_j(\mu)\{x^T[P(A_i + B_iK_j) \\ &\quad + (A_i + B_iK_j)^T P]x - 2x^T PB_{eij}\bar{e} \\ &\quad + 2x^T PB_{di}d + x^T W_i^T PW_ix+ \\ &\quad \bar{e}^T(\bar{P}R_i + R_i^T \bar{P})\bar{e} - 2\bar{e}^T \bar{P}L_{si}u_s \\ &\quad + 2\bar{e}^T \bar{P}TJ_1\ddot{f} + x^T \bar{W}_i^T T^T \bar{P}T\bar{W}_ix\} \end{aligned} \tag{35}$$

$s \in \mathcal{R}^{l_f}$, it can be noticed that $s^T sgn(s) = \Xi^{l_f}_{i=1}|s_i|$ and $\|s\| \le \Xi^{l_f}_{i=1}|s_i|$.

If we choose $L_{si} = TJ_1$, we can obtain

$$\begin{aligned} &= \Xi^r_{i=1}h_i(\mu)(-2\bar{e}^T \bar{P}L_{si}u_s + 2\bar{e}^T \bar{P}TJ_1\ddot{f}) \\ &= -2\bar{e}^T \bar{P}TJ_1(\delta + \varepsilon)sgn(s) + 2\bar{e}^T \bar{P}TJ_1\ddot{f} \\ &= 2s^T \ddot{f} - 2s^T(\delta + \varepsilon)sgn(s) \\ &\le 2\|s\|\|\ddot{f}\| - 2(\delta + \varepsilon)\|s\| \\ &\le -2\varepsilon\|s\| \end{aligned} \tag{36}$$

Hence it is shown that when $d = 0$

$$\begin{aligned} \mathcal{L}V \le &\Xi^r_{i=1}h_i(\mu)\Xi^r_{j=1}h_j(\mu)\{x^T[P(A_i + B_iK_j) \\ &+ (A_i + B_iK_j)^T P]x - 2x^T PB_{eij}\bar{e} + x^T W_i^T PW_ix \\ &+ \bar{e}^T(\bar{P}R_i + R_i^T \bar{P})\bar{e} + x^T \bar{W}_i^T T^T \bar{P}T\bar{W}_ix - 2\varepsilon\|s\|\} \\ \le &\Xi^r_{i=1}h_i(\mu)\Xi^r_{j=1}h_j(\mu)\{[x^T \;\; \bar{e}^T]\Pi_{ij}\begin{bmatrix} x \\ \bar{e} \end{bmatrix}\} - 2\varepsilon\|s\| \end{aligned} \tag{37}$$

where $\Pi_{ij} = \begin{bmatrix} \Pi_{ij11} & -PB_{eij} \\ * & \Pi_{ij22} \end{bmatrix}, \Pi_{ij11} = P(A_i+B_iK_j)+(A_i + B_iK_j)^T P+W_i^T PW_i+\bar{W}_i^T T^T \bar{P}T\bar{W}_i$, and $\Pi_{ij22} = \bar{P}T\bar{A}_i+\bar{A}_i^T T^T \bar{P}-\bar{P}L_{i1}\bar{C}-\bar{C}^T L_{i1}^T \bar{P}$. From LMIs (29), we can see $\begin{bmatrix} \Pi_{ij11} + C^T C & PB_{di} & -PB_{eij} + C^T D_e \\ * & -\gamma^2 I_{ld} & 0 \\ * & * & \Pi_{ij22} + D_e^T D_e \end{bmatrix} < 0$

which means

$$\begin{bmatrix} \Pi_{ij11} + C^T C & -PB_{eij} + C^T D_e & PB_{di} \\ * & \Pi_{ij22} + D_e^T D_e & 0 \\ * & * & -\gamma^2 I_{ld} \end{bmatrix} < 0$$

leading to

$$\begin{bmatrix} \Pi_{ij11} + C^T C & -PB_{eij} + C^T D_e \\ * & \Pi_{ij22} + D_e^T D_e \end{bmatrix} < 0.$$

Then

$$\begin{bmatrix} \Pi_{ij11} & -PB_{eij} \\ * & \Pi_{ij22} \end{bmatrix} + \begin{bmatrix} C^TC & C^TD_e \\ * & D_e^TD_e \end{bmatrix} < 0.$$

Since

$$\begin{bmatrix} C^TC & C^TD_e \\ * & D_e^TD_e \end{bmatrix} = \begin{bmatrix} C^T \\ D_e^T \end{bmatrix} \begin{bmatrix} C & D_e \end{bmatrix} > 0$$

We have

$$\begin{bmatrix} \Pi_{ij11} & -PB_{eij} \\ * & \Pi_{ij22} \end{bmatrix} < 0.$$

Hence we have $\Pi_{ij} < 0$, indicating a positive scalar c_3 can be found such that

$$\mathcal{E}(\mathcal{L}V) < -c_3\mathcal{E}(\|\tilde{x}\|^2) \tag{38}$$

According to Lemma 1, system (27) is stochastically exponentially stable in mean square. When $d \neq 0$, let us move on to discuss the robustness of the system against unknown inputs. Consider the following performance index:

$$\Gamma = \mathcal{E}\{\int_0^{Tf} [y_c^T(\tau)y_c(\tau) - \gamma^2 d^T(\tau)d(\tau)]d\tau\} \tag{39}$$

Then adding and subtracting $\mathcal{E}(\int_0^{Tf} \mathcal{L}V d\tau)$ to (39) yields:

$$\begin{aligned}\Gamma \leq \mathcal{E}\{&\int_0^{Tf} \Xi_{i=1}^r h_i(\mu)\Xi_{j=1}^r h_j(\mu)[x^T \quad d^T \quad \bar{e}^T]\Omega_{ij} \begin{bmatrix} x \\ d \\ \bar{e} \end{bmatrix} d\tau\} \\ &- \mathcal{E}(\int_0^{Tf} \mathcal{L}V d\tau)\end{aligned} \tag{40}$$

where $\Omega_{ij} = \begin{bmatrix} \Omega_{ij11} & PB_{di} & -PB_{eij} + C^TD_e \\ * & -\gamma^2 I_{ld} & 0 \\ * & * & \Omega_{ij33} \end{bmatrix}$, $\Omega_{ij11} = P(A_i+B_iK_j)+(A_i + B_iK_j)^TP+W_i^TPW_i+\bar{W}_i^TT^T\bar{P}T\bar{W}_i+C^TC$, and $\Omega_{ij33} = \bar{P}T\bar{A}_i+\bar{A}_i^TT^T\bar{P}-\bar{P}L_{i1}\bar{C}-\bar{C}^TL_{i1}^T\bar{P}+D_e^TD_e$.

It is not hard to find $\mathcal{E}(\int_0^{Tf} \mathcal{L}V d\tau) = \mathcal{E}(V) > 0$. Thus if $\Omega_{ij} < 0$, we get $\Gamma < 0$, leading to $\mathcal{E}(\|y_c\|_{Tf}^2) < \gamma^2\mathcal{E}(\|d\|_{Tf}^2)$.

As a results, LMIs (29) can guarantee the stochastically exponentially stability in mean square of system (27) and satisfy performance $\mathcal{E}(\|y_c\|_{Tf}^2) < \gamma^2\mathcal{E}(\|d\|_{Tf}^2)$. This completes the proof.

It is noticed that both the system dynamics and error dynamics are subject to state Brownian fluctuation, which makes it challenging to design observer and controller gains simultaneously. In order to simplify the challenging matrix problem, the following theorem is proposed.

Theorem 2. The closed-loop system (27) is stochastic stable in mean square and satisfy $\mathcal{E}(\|y_c\|_{Tf}^2) < \gamma^2\mathcal{E}(\|d\|_{Tf}^2)$, if $\forall i, j$

(1) There are positive matrix P, matrices K_j, and positive scalars σ and γ_1, where $\gamma_1 < \gamma$ such that

$$\begin{bmatrix} \Lambda_{ij} & PB_{di} \\ * & -\gamma^2 I_{ld} \end{bmatrix} < \begin{bmatrix} -\sigma PP & 0 \\ * & -\gamma_1^2 I_{ld} \end{bmatrix} \tag{41}$$

where $\Lambda_{ij} = P(A_i + B_iK_j) + (A_i + B_iK_j)^TP + W_i^TPW_i + C^TC, i, j = 1, 2, ...r.$

(2) There are positive matrix $\bar{P}$, matrices L_{i1}, N such that

$$\begin{bmatrix} -\sigma PP + \bar{W}_i^TT^T\bar{P}T\bar{W}_i & 0 & -PB_{eij} + C^TD_e \\ * & -\gamma_1^2 I_{ld} & 0 \\ * & * & \Omega_{ij33} \end{bmatrix} < 0 \tag{42}$$

and

$$J_1^TT^T\bar{P} = N\bar{C} \tag{43}$$

where P and K_j are obtained from solving (41) , $i, j = 1, 2, ...r.$

Proof. According to inequality (41), $\forall i, j$, $\begin{bmatrix} -\sigma PP & 0 \\ * & -\gamma_1^2 I_{ld} \end{bmatrix} + \Theta < 0$ implies that $\begin{bmatrix} \Lambda_{ij} & PB_{di} \\ * & -\gamma^2 I_{ld} \end{bmatrix} + \Theta < 0$, where Θ

is a semi-positive symmetric matrix. Therefore,

$$\begin{aligned}
\Omega_{ij} &= \begin{bmatrix} \Lambda_{ij} & PB_{di} & 0 \\ * & -\gamma^2 I_{ld} & 0 \\ * & * & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \bar{W}_i^T T^T \bar{P} T \bar{W}_i & 0 & -PB_{eij} + C^T D_e \\ * & 0 & 0 \\ * & * & \Omega_{ij33} \end{bmatrix} \\
&\leq \begin{bmatrix} -\sigma PP & 0 & 0 \\ * & -\gamma_1^2 I_{ld} & 0 \\ * & * & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \bar{W}_i^T T^T \bar{P} T \bar{W}_i & 0 & -PB_{eij} + C^T D_e \\ * & 0 & 0 \\ * & * & \Omega_{ij33} \end{bmatrix} \\
&= \begin{bmatrix} -\sigma PP + \bar{W}_i^T T^T \bar{P} T \bar{W}_i & 0 & -PB_{eij} + C^T D_e \\ * & -\gamma_1^2 I_{ld} & 0 \\ * & * & \Omega_{ij33} \end{bmatrix}
\end{aligned} \tag{44}$$

So (41) and (42) indicate $\Omega_{ij} < 0$, which meets LMI (29). This completes the poof.

In this way, a sequential design approach can be obtained to reduce the complication for solving observer and controller gains by Theorem 2. Nevertheless, it can be noticed that LMIs (41) and (42) are nonlinear. In order to transform it into linear range, the following theorem is proposed.

Theorem 3. The closed loop system (27) are stochastically stable in mean square and satisfy $\mathcal{E}(\|y_c\|_{Tf}^2) < \gamma^2 \mathcal{E}(\|d\|_{Tf}^2)$, if $\forall i, j$

(1) There are positive matrix P, matrices K_j, and positive scalars σ and γ_1, where $\gamma_1 < \gamma$ such that

$$\begin{bmatrix} \Psi_{ij} & XW_i^T & XC^T & PB_{di} \\ * & -X & 0 & 0 \\ * & * & -I_n & 0 \\ * & * & * & (\gamma_1^2 - \gamma^2) I_{ld} \end{bmatrix} < 0 \tag{45}$$

where $X = P^{-1}, \Psi_{ij} = A_i X + X A_i + B_i Y_j + Y_i^T B_i^T + \sigma I_n, Y_j = K_j X$. Then the control gains can be selected as $K_j = Y_j X^{-1}$.

(2) There are positive matrix $\bar{P}$, matrices Q_i, N such that

$$\begin{bmatrix} -\sigma PP + \bar{W}_i^T T^T \bar{P} T \bar{W}_i & 0 & -PB_{eij} + C^T D_e \\ * & -\gamma_1^2 I_{ld} & 0 \\ * & * & \Phi_{ij33} \end{bmatrix} < 0 \tag{46}$$

and

$$J_1^T T^T \bar{P} = N \bar{C} \tag{47}$$

where $\Phi_{ij33} = \bar{P} T \bar{A}_i + \bar{A}_i^T T^T \bar{P} - Q_i \bar{C} - \bar{C}^T Q_i^T + D_e^T D_e$. And the observer gains $L_{i1} = \bar{P}^{-1} Q_i$.

Proof. Inequality (41) can be rewritten as:

$$\begin{bmatrix} \Lambda_{ij} + \sigma PP & PB_{di} \\ * & (\gamma_1^2 - \gamma^2) I_{ld} \end{bmatrix} < 0 \tag{48}$$

Multiplying $\begin{bmatrix} P^{-T} & 0 \\ 0 & I_{ld} \end{bmatrix}$ on the left side and $\begin{bmatrix} P^{-1} & 0 \\ 0 & I_{ld} \end{bmatrix}$ on the right side of (48) we can have

$$\begin{bmatrix} \Psi_{ij} + XW_i^T X^{-1} W_i X + XC^T CX & PB_{di} \\ * & (\gamma_1^2 - \gamma^2) I_{ld} \end{bmatrix} < 0 \tag{49}$$

where $X = P^{-1}, \Psi_{ij} = A_i X + X A_i + B_i Y_j + Y_i^T B_i^T + \sigma I_n, Y_j = K_j X$. Based on Schur complement, (49) is equivalent to

$$\begin{bmatrix} \Psi_{ij} & XW_i^T & XC^T & PB_{di} \\ * & -X & 0 & 0 \\ * & * & -I_n & 0 \\ * & * & * & (\gamma_1^2 - \gamma^2) I_{ld} \end{bmatrix} < 0 \tag{50}$$

By using $Q_i = \bar{P} L_{i1}$, (42) can be rewritten as

$$\begin{bmatrix} -\sigma PP + \bar{W}_i^T T^T \bar{P} T \bar{W}_i & 0 & -PB_{eij} + C^T D_e \\ * & -\gamma_1^2 I_{ld} & 0 \\ * & * & \Phi_{ij33} \end{bmatrix} < 0 \tag{51}$$

This completes the proof.

Until now, sufficient conditions have been proposed for stochastically exponentially stability in mean square of system (27). However, (47) in Theorem 3 is a matrix equality which is difficulty to be solved by available software toolbox directly. In order to simplify this problem, equation (47) can be rewritten as

$$J_1^T T^T \bar{P} = N \bar{C} \tag{52}$$

which can also be represented by

$$trace\{(J_1^T T^T \bar{P} - N\bar{C})^T (J_1^T T^T \bar{P} - N\bar{C})\} = 0 \quad (53)$$

Then we introduce the following condition:

$$(J_1^T T^T \bar{P} - N\bar{C})^T (J_1^T T^T \bar{P} - N\bar{C}) < \theta I \quad (54)$$

where θ is a positive scalar. According to Schur complement, (54) is equivalent with:

$$\begin{bmatrix} -\theta I & (J_1^T T^T \bar{P} - N\bar{C})^T \\ (J_1^T T^T \bar{P} - N\bar{C}) & -I \end{bmatrix} < 0 \quad (55)$$

As a result, condition (45), (46) and (47) can be converted into searching a global solution of the following problem:

$$min\ \theta, subject\ to\ (45), (46), and\ (55) \quad (56)$$

which can be solved by employing Solvers *mincx* in LMI toolbox of Matlab.

4.2 Reachability of the sliding mode surface

The above sections present the approaches to design robust estimator-based fault tolerant control scheme. Now let us look at the reachability of the sliding mode surface. It is known from [23] that the solution of $\bar{e}$ is given by

$$\begin{aligned} \bar{e} = \Xi_{i=1}^r h_i(\mu)\{ &\int_0^t [R_i\bar{e}(\tau) - L_{si}u_s(\tau) + TJ_1\ddot{f}(\tau)]d\tau \\ &+ \int_0^t T\bar{W}_i x(\tau) d\omega(\tau)\} \end{aligned} \quad (57)$$

Substituting it into (19), we have

$$\begin{aligned} s = \Xi_{i=1}^r h_i(\mu)\{ &\int_0^t J_1^T T^T \bar{P}[R_i\bar{e}(\tau) - L_{si}u_s(\tau) \\ &+ TJ_1\ddot{f}(\tau)]d\tau + \int_0^t J_1^T T^T \bar{P} T\bar{W}_i x(\tau) d\omega(\tau)\} \end{aligned} \quad (58)$$

If the following condition holds:

$$J_1^T T^T \bar{P} T \bar{W}_i = 0 \quad (59)$$

we can yield

$$\begin{aligned} s = \Xi_{i=1}^r h_i(\mu) &\int_0^t J_1^T T^T \bar{P}[R_i\bar{e}(\tau) - L_{si}u_s(\tau) \\ &+ TJ_1\ddot{f}(\tau)]d\tau \end{aligned} \quad (60)$$

which implies

$$\dot{s} = \Xi_{i=1}^r h_i(\mu) J_1^T T^T \bar{P}(R_i\bar{e} - L_{si}u_s + TJ_1\ddot{f}) \quad (61)$$

Remark 3. From condition (59), the stochastic influences have been removed from the sliding mode surface, which means $s(t) = 0$ is not stochastic.

In order to satisfy equation (58), which is equivalent to

$$trace\{(J_1^T T^T \bar{P} T\bar{W}_i)^T (J_1^T T^T \bar{P} T\bar{W}_i)\} = 0 \quad (62)$$

By employing the same parameter θ in (54), we have

$$(J_1^T T^T \bar{P} T\bar{W}_i)^T (J_1^T T^T \bar{P} T\bar{W}_i) < \theta I \quad (63)$$

Applying Schur complement, (63) can be rewritten as:

$$\begin{bmatrix} -\theta I & (J_1^T T^T \bar{P} T\bar{W}_i)^T \\ J_1^T T^T \bar{P} T\bar{W}_i & -I \end{bmatrix} < 0 \quad (64)$$

Then equality (59) can also be converted into a minimization problem as

$$min\ \theta, subject\ to\ (64) \quad (65)$$

Remark 4. It is shown that both specified sliding surface and desired sliding mode control law can be constructed via convex optimization problem, which can be handled by LMI toolbox of Matlab.

Finally, we can generate the following theorem to design observer and controller gains.

Theorem 4. For system (2), there exists a tolerant observer-based controller in the form of (10), (22) and (25), ensuring that

(i)The closed-loop system (27) is stochastically exponentially stable in mean square ;

(ii)The error trajectories $\bar{e}$ can be globally driven onto the sliding surface $s(t) = 0$ in probability;

(iii)The compensated output satisfies $\mathcal{E}(\|y_c\|_{Tf}^2) < \gamma^2 \mathcal{E}(\|d\|_{Tf}^2)$

if $\forall i, j$, we can find a global solution of the following optimization problem:

$$min\ \theta, subject\ to\ (45), (46), (55)\ and\ (64) \quad (66)$$

Proof. According to Theorem 3, conditions (i) and (iii) can be guaranteed by optimization problem (66). Now let us prove condition (ii). Consider the following Lyapunov candidate for sliding mode surface $s(t)=0$:

$$V_3 = \frac{1}{2}s^T(J_1^T T^T \bar{P} T J_1)^{-1} s \tag{67}$$

Its derivative can be obtained as:

$$\begin{aligned}\dot{V}_3 = \Xi_{i=1}^r h_i(\mu) s^T(J_1^T T^T \bar{P} T J_1)^{-1} J_1^T T^T \bar{P}(R_i\bar{e} \\ - L_{si}u_s + TJ_1\ddot{f})\end{aligned} \tag{68}$$

It can be calculated that

$$\begin{aligned}&\Xi_{i=1}^r h_i(\mu) s^T(J_1^T T^T \bar{P} T J_1)^{-1} J_1^T T^T \bar{P}(-L_{si}u_s + TJ_1\ddot{f}) \\ &= s^T(J_1^T T^T \bar{P} T J_1)^{-1} J_1^T T^T \bar{P} T J_1[-(\delta+\varepsilon)sgn(s)+\ddot{f}] \\ &= s^T[-(\delta+\varepsilon)sgn(s)+\ddot{f}] \\ &\leq -(\delta+\varepsilon)\|s\| + \|s\|\|\ddot{f}\| \\ &\leq -\varepsilon\|s\|\end{aligned} \tag{69}$$

Then we have

$$\begin{aligned}\dot{V}_3 &\leq \Xi_{i=1}^r h_i(\mu)[s^T(J_1^T T^T \bar{P} T J_1)^{-1}(J_1^T T^T \bar{P} R_i \bar{e})] - \varepsilon\|s\| \\ &\leq \Xi_{i=1}^r h_i(\mu)\|s\|[\|Z_i\|\mathcal{E}(\|\bar{e}\|) - \varepsilon]\end{aligned} \tag{70}$$

where $Z_i = (J_1^T T^T \bar{P} T J_1)^{-1}(J_1^T T^T \bar{P} R_i)$.

Define the following variables:

$$\rho(h_i(\mu)) = \Xi_{i=1}^r h_i(\mu)\|Z_i\| \tag{71}$$

It is easy to find the upper bounds ρ_0 to make $\rho(h_i(\mu)) < \rho_0$. Hence

$$\begin{aligned}\dot{V}_3 &< \rho_0\|s\|\mathcal{E}(\|\bar{e}\|) - \varepsilon\|s\| \\ &= -\|s\|[\varepsilon - \rho_0\mathcal{E}(\|\bar{e}\|)]\end{aligned} \tag{72}$$

We define the following domain:

$$\Omega(\rho_0) = \{\mathcal{E}(\|\bar{e}\|) < \tfrac{\varepsilon}{\rho_0}\} \tag{73}$$

then

$$\dot{V}_3 < 0 \tag{74}$$

Since the error and original systems are stable, the trajectory of $\bar{e}$ enter in $\Omega(\rho_0)$ remains there, which means condition (ii) is satisfied by optimization problem (66).

4.3 Design procedures of fault estimator-based fault tolerant control

On the basis of the above theorems, the design procedures of integrated observer-based fault estimation and the corresponding fault tolerant control can be summarized as follows:

(1) Construct the augmented system in the form of (8) for stochastic T-S fuzzy systems in presence of faults and unknown inputs.

(2) Select matrix H^* in the form of (17), and T can be yielded as $T = I_{\bar{n}} - H\bar{C}$.

(3) Solve LMIs (45) to obtain matrices X and Y_j, and calculate $P = X^{-1}, K_j = Y_j X^{-1}$.

(4) Select the sliding mode control gains as $L_{si} = TJ_1$.

(5) Solve optimization problem

$$min\ \theta, subject\ to\ (46), (55)\ and\ (64)$$

by submitting P and K_j to obtain matrices $\bar{P}, Q_i, N$, and the observer gains can be derived as $L_{i1} = \bar{P}^{-1}Q_i$.

(6) Calculate the other observer gains R_i and L_{i2} following the formulas (13) and (15), respectively.

(7) Implement the robust sliding mode unknown input observer (10) with the sliding term $u_s = (\delta+\varepsilon)sgn(s)$, where $s = Ny - N\bar{C}\hat{\bar{x}}$, and obtain the augmented estimate $\hat{\bar{x}}$, leading to simultaneous estimates of system states and faults $\hat{x}$ and $\hat{f}$.

(8) Implement control law $u = \Xi_{j=1}^r h_j(\mu)\bar{K}_j\hat{\bar{x}}$ and sensor compensation output $y_c = y - D_f\hat{f}$, where $\bar{K}_j = [K_j \ \ 0 \ \ K_f]$ and $K_f = -B_h^+ B_{fh}$.

Remark 5. In the designed sliding mode unknown input observer, the unknown inputs can be decoupled by the designing of observer gain H. It is noticed that the Assumption (1) should be satisfied for such an observer. If this condition cannot be satisfied, which means not all disturbances can be decoupled,then $\forall i$, we can let $B_{Ji} = [\bar{B}_{dui} \ \ J_1]$ to replace J_1, and $d_f = [d \ \ \ddot{f}]^T$ to replace $\ddot{f}$ in augmented system (8),where$\bar{B}_{dui} = \begin{cases} 0^{\bar{n}\times l_d} & rank(\bar{C}\bar{B}_{di}) = rank(\bar{B}_{di}) \\ \bar{B}_{di} & rank(\bar{C}\bar{B}_{di}) \neq rank(\bar{B}_{di}) \end{cases}$. Then the sliding mode item can attenuate the unknown inputs that cannot be decoupled by UIO. Alternatively, recent development about cascade observer (see [3]) and high-order sliding mode observer (see [2]) is potential to relax this condition, which is of interest in the future work.

5 Extension to stochastic fuzzy system with uncertainties and faults in stochastic perturbation.

Consider a stochastic T-S fuzzy system with uncertainties and faults in stochastic perturbation described by:

$$\begin{cases} dx = \Xi_{i=1}^{r} h_i(\mu)[(A_i x + B_i u + B_{di} d + B_{fi} f)dt \\ \qquad + (W_i x + G_i u + G_{fi} f + M_i d_2) d\omega] \\ y = Cx + D_f f \end{cases} \tag{75}$$

where $d_2 \in \mathcal{R}^{l_{d2}}$ represents unknown inputs on stochastic perturbation; $\forall i, G_i = \beta_s B_i, G_{fi} = \beta_{sf} B_{fi}$ and M_i are coefficients of faults and unknown inputs on stochastic perturbation, respectively, where β_s and β_{sf} are positive scalars. Other symbols are with the same meaning in the previous sections. Then the following augmented plant can be constructed for (75).

$$\begin{cases} d\bar{x} = \Xi_{i=1}^{r} h_i(\mu)[(\bar{A}_i \bar{x} + \bar{B}_i u + \bar{B}_{di} d + J_1 \ddot{f})dt \\ \qquad + (\bar{W}_i x + \bar{G}_i u + \bar{G}_{fi} f + \bar{M}_i d_2) d\omega] \\ y = \bar{C}\bar{x} \end{cases} \tag{76}$$

where

$$\begin{aligned} \bar{G}_i &= [G_i^T \;\; 0_{m \times l_f} \;\; 0_{m \times l_f}]^T \in \mathcal{R}^{\bar{n} \times m}, \\ \bar{G}_{fi} &= [G_{fi}^T \;\; 0_{l_f \times l_f} \;\; 0_{l_f \times l_f}]^T \in \mathcal{R}^{\bar{n} \times l_f}, \\ \bar{M}_i &= [M_i^T \;\; 0_{l_{d2} \times l_f} \;\; 0_{l_{d2} \times l_f}]^T \in \mathcal{R}^{\bar{n} \times l_{d2}}, \end{aligned}$$

Implementing observer (10), controller (22) and (25), we can obtain the following closed-loop system:

$$\begin{cases} dx = \Xi_{i=1}^{r} h_i(\mu) \Xi_{j=1}^{r} h_j(\mu) \{[(A_i + B_i K_j)x - B_{eij}\bar{e} \\ \qquad + B_{di} d]dt + [(W_i + G_i K_j)x - G_{eij}\bar{e} + M_i d_2] d\omega\} \\ d\bar{e} = \Xi_{i=1}^{r} h_i(\mu)[(R_i \bar{e} - L_{si} u_s + T J_1 \ddot{f})dt \\ \qquad + [T(\bar{W}_i + \bar{G}_i K_j)x - T\bar{G}_{eij}\bar{e} + T\bar{M}_i d_2] d\omega] \\ y = Cx + D_e \bar{e} \end{cases} \tag{77}$$

where $G_{eij} = G_i K_j J_0 - G_{fi} J_2$ and $\bar{G}_{eij} = \bar{G}_i K_j J_0 - \bar{G}_{fi} J_2$.

Similar with the deviation in Theorem 4, we have the following theorem readily

Theorem 5. For system (75), there exists a tolerant observer-based controller in the form of (10), (22) and (25), ensuring that

(i) The closed-loop system (77) is stochastically exponentially stable in mean square ;

(ii) The error trajectories $\bar{e}$ can be globally driven onto the sliding surface $s(t) = 0$ in probability;

(iii) The compensated output satisfies $\mathcal{E}(\|y_c\|_{Tf}^2) < \gamma^2 \mathcal{E}(\|d\|_{Tf}^2) + \gamma_2^2 \mathcal{E}(\|d_2\|_{Tf}^2)$; If we can find a global solution of the following optimization problem

$$min\ \theta, subject\ to$$

$$\begin{bmatrix} \Psi_{ij} & XW_i^T + Y_j^T G_i^T & XC^T & PB_{di} \\ * & -X & 0 & 0 \\ * & * & -I_n & 0 \\ * & * & * & (\gamma_1^2 - \gamma^2) I_{ld} \end{bmatrix} < 0 \tag{78}$$

$$\begin{bmatrix} \Theta_{ij11} & 0 & \Theta_{ij13} & \Theta_{ij14} \\ * & -\gamma_1^2 I_{ld} & 0 & 0 \\ * & * & \Theta_{ij33} & \Theta_{ij34} \\ * & * & * & \Theta_{ij44} \end{bmatrix} < 0 \tag{79}$$

$$\begin{bmatrix} -\theta I & (J_1^T T^T \bar{P} - N\bar{C})^T \\ (J_1^T T^T \bar{P} - N\bar{C}) & -I \end{bmatrix} < 0 \tag{80}$$

$$\begin{bmatrix} -\theta I & (J_1^T T^T \bar{P} T \bar{W}_i)^T \\ J_1^T T^T \bar{P} T \bar{W}_i & -I \end{bmatrix} < 0 \tag{81}$$

where P and $\bar{P}$ are positive matrices, σ, γ_1,and γ_2 are positive scalars where $\gamma_1 < \gamma$.

$X = P^{-1}$,

$\Psi_{ij} = A_i X + X A_i + B_i Y_j + Y_i^T B_i^T + \sigma I_n$,

$Y_j = K_j X$,

$\Theta_{ij11} = -\sigma PP + (\bar{W}_i + \bar{G}_i K_j)^T T^T \bar{P} T (\bar{W}_i + \bar{G}_i K_j)$,

$\Theta_{ij13} = -PB_{eij} - (W_i + G_i K_j)^T P G_{eij} + (\bar{W}_i + \bar{G}_i K_j)^T T^T \bar{P} T \bar{G}_{eij} + C^T D_e$,

$\Theta_{ij33} = \bar{P} T \bar{A}_i + \bar{A}_i^T T^T \bar{P} - Q_i \bar{C} - \bar{C}^T Q_i^T + G_{eij}^T P G_{eij} + \bar{G}_{eij}^T T^T \bar{P} T \bar{G}_e ij + D_e^T D_e$,

$\Theta_{ij14} = (W_i + G_i K_j)^T P M_i + (\bar{W}_i + \bar{G}_i K_j)^T T^T \bar{P} T \bar{M}_i$,

$\Theta_{ij34} = -G_{eij}^T P M_i - \bar{G}_{eij}^T T^T \bar{P} T \bar{M}_i$,

$\Theta_{ij44} = M_i^T P M_i + \bar{M}_i^T T^T \bar{P} T \bar{M}_i - \gamma_2^2 I_{ld2}$.

Then the control gains can be selected as $K_j = Y_j X^{-1}$. And the observer gains $L_{i1} = \bar{P}^{-1} Q_i$.

6 Simulation study

6.1 Example 1. A single-link manipulator

In this section, the developed technique is applied to single-link manipulator with revolute joints actuated by a DC motor ([14], [20]). When the motor is working, many environmental factors such as the temperature, the random vibrate friction coefficient for some materials, the stochastic changes in the load and so on, may introduce random disturbances to the rotor ([20]). Then the plant can be described by the following stochastic nonlinear system

$$\begin{cases} d\theta_m = (w_m + \alpha_{d1} d)dt + W_{12} w_m d\omega \\ dw_m = [\dfrac{k}{J_m}(\theta_l - \theta_m) - \dfrac{G}{J_m} w_m + \dfrac{k_\tau}{J_m} u + \alpha_{d2} d]dt \\ \qquad + (W_{23}\theta_l + W_{21}\theta_m + W_{22} w_m)d\omega \\ d\theta_l = (w_l + \alpha_{d3} d)dt + W_{34} w_l d\omega \\ dw_l = [-\dfrac{k}{J_l}(\theta_l - \theta_m) - \dfrac{mgh}{J_l} sin(\theta_l) + \alpha_{d4} d]dt \\ \qquad + (W_{43}\theta_l + W_{41}\theta_m + W_s sin(\theta_l))d\omega \end{cases} \tag{82}$$

where J_m represents the inertia of the DC motor, J_l is the inertia of the link, θ_m and θ_l denote the angles of the rotations of the motor and link, respectively, w_m and w_l are the angular velocities of the motor and link, respectively, k is torsional spring constant, k_τ is the amplifier gain, G is the viscous friction, m is the pointer mass, g is the gravity constant, and h is the distance from the rotor to the gravity center of the link, and u is the control input (D-C voltage) to produce the motor torque. $W_{12} = 0.0006, W_{23} = 0.001, W_{21} = -0.002, W_{22} = -0.0001, W_{34} = 0.0005, W_{43} = 0.006, W_{41} = -0.0004, W_s = 0.003$ are coefficients of stochastic parameter perturbations. $\alpha_{d1} = 0.1, \alpha_{d2} = -0.2, \alpha_{d3} = 0.3, \alpha_{d4} = 0.5$ are coefficients of deterministic extra unknown input signal d.

Letting $x = [\theta_m \quad w_m \quad \theta_l \quad w_l]$, with the system can be written in the form of (1), where the system coefficients are given as follows ([15], [17]):

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -22.83 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -18.77 & 0 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, B_d = \begin{bmatrix} 0.1 \\ -0.2 \\ 0.3 \\ 0.5 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0 & 0.0006 & 0 & 0 \\ -0.002 & -0.0001 & 0.001 & 0 \\ 0 & 0 & 0 & 0.0005 \\ -0.0004 & 0 & 0.009 & 0 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0 & 0.0006 & 0 & 0 \\ -0.002 & -0.0001 & 0.001 & 0 \\ 0 & 0 & 0 & 0.0005 \\ -0.0004 & 0 & 0.005 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

with initial condition to be $x_0 = [0.1 \quad 0.05 \quad 0.02 \quad 0.03]^T$ corrupted by random noises. And the membership functions are $h_1 = \frac{\mu(t)+0.2172}{1.217}, h_2 = \frac{1-\mu(t)}{1.217}$, where $\mu(t) = \frac{sin(\theta_l)}{\theta_l}$.

The actuator fault f_a taken into account is 40% loss of actuation effectiveness from 2.5 sec. to 5 sec. and the sensor fault concerned f_s is stuck fault from 7 sec. to 10 sec., located in the first output sensor. Then $B_{f_{a1}} = B_1, B_{f_{a2}} = B_2$, and $D_{f_s} = [1 \quad 0 \quad 0]^T$. By representing $f = [f_a^T \quad f_s^T]^T$, we have $B_{f1} = [B_{f_{a1}} \quad 0_{4\times 1}], B_f 2 = [B_{f_{a2}} \quad 0_{4\times 1}]$ and $D_f = [0_{3\times 1} \quad D_{f_s}]$. Unknown input signal d is supposed to be random noises from -0.1 and 0.1. The reference input is given as $u_r = 2$. Observer gain H can be solved by (17) as

$$H = \begin{bmatrix} 0.0714286 & -0.1428571 & 0.2142857 \\ -0.1428571 & 0.2857143 & -0.4285714 \\ 0.2142857 & -0.4285714 & 0.6428571 \\ 0.3571429 & -0.7142857 & 1.0714286 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Choosing $\gamma = 2, \gamma_1 = 0.01, \sigma = 0.2$, and solving LMI problem (66), the controller and observer gains can be calculated as follows:

$$K_1 = K_2 = [0.6719 \quad -0.4519 \quad -1.4341 \quad -0.0296)],$$

$$
L_{11} = 10^4 \times \begin{bmatrix} -8.8150386 & 3.3295948 & -2.4773194 \\ 46.891958 & 44.797501 & -27.764813 \\ -31.185141 & 1.7156617 & -2.1304054 \\ -45.433532 & 14.560286 & -11.136679 \\ 11187.883 & 35138.110 & -21745.916 \\ 203160.41 & 68559.602 & -27995.815 \\ 346.91612 & 2753.7331 & -1985.0225 \\ 196.15624 & 81.132301 & -46.652256 \end{bmatrix},
$$

$$
L_{21} = 10^4 \times \begin{bmatrix} -8.8150344 & 3.3296205 & -2.4772679 \\ 46.891956 & 44.797756 & -27.764238 \\ -31.185130 & 1.7156837 & -2.1303699 \\ -45.433510 & 14.560403 & -11.136044 \\ 11187.891 & 35138.335 & -21745.424 \\ 203160.37 & 68560.030 & -27994.798 \\ 346.91677 & 2753.7509 & -1984.9846 \\ 196.15620 & 81.132718 & -46.651262 \end{bmatrix},
$$

and we can obtain $\theta = 10^{-9}$, which means conditions $J_1^T T^T \bar{P} = N\bar{C}$ and $J_1^T T^T \bar{P} T \bar{W}_i = 0$ hold. Observer gains K_{f1} and K_{f2} can be calculated according to (23). The sliding mode controller gain can be obtained as $L_{si} = TJ_1$. Other observer gains T, L_{12}, L_{22}, R_1 and R_2 can be calculated according to (13) to (15) respectively. Select $\rho_0 = 0.00001$, and $\varepsilon = 0.00001$, and using the Euler-Maruyama method ([12]) to simulate the standard Brownian motions (with 5 Brownian paths), one can obtain the simultaneous estimation results of full system states and concerned faults as shown in Figs. 2-7. Estimation of states in all five paths can be generated. In the results, the means of state estimates are shown to compare with original states. Figs. 8-10 compare the outputs with and without signal compensations, and outputs in fault-free case.

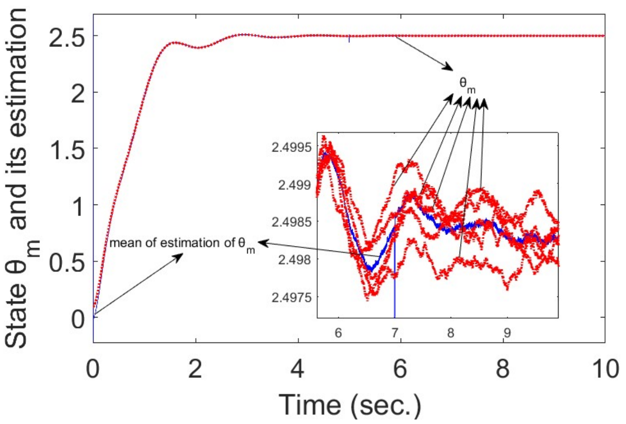

Fig. 2. θ_m and the mean of its estimation.

From Figs 2-7, the estimation performance of both the system states and considered faults are satisfactory. Moreover, we can find in Figs 8-10 that the actuator and sensor faults will make the deviation of the outputs. However, after tolerant control, the deviation is eliminated/offset successfully, as one can see the compensated outputs are consistent with the fault-free outputs.

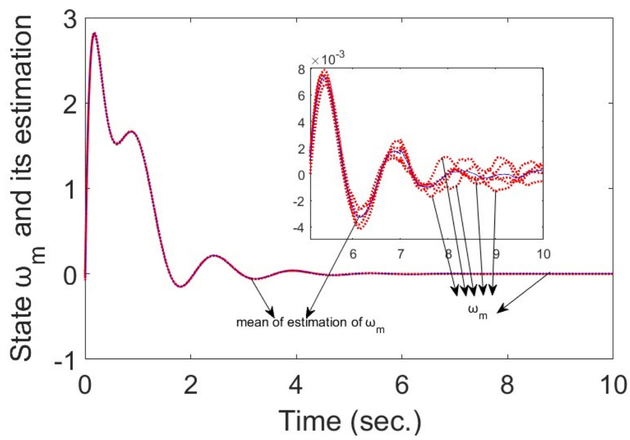

Fig. 3. w_m and the mean of its estimation.

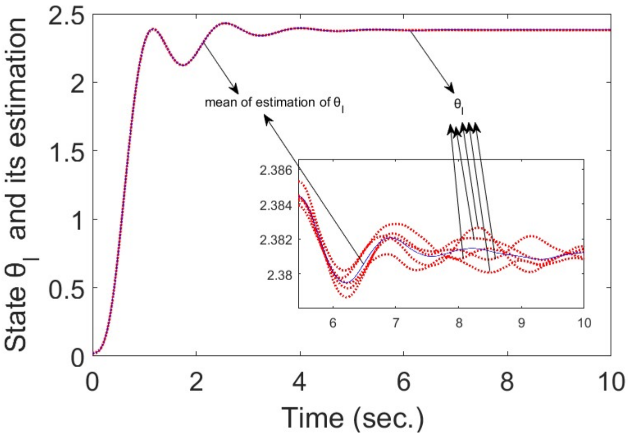

Fig. 4. θ_l and the mean of its estimation.

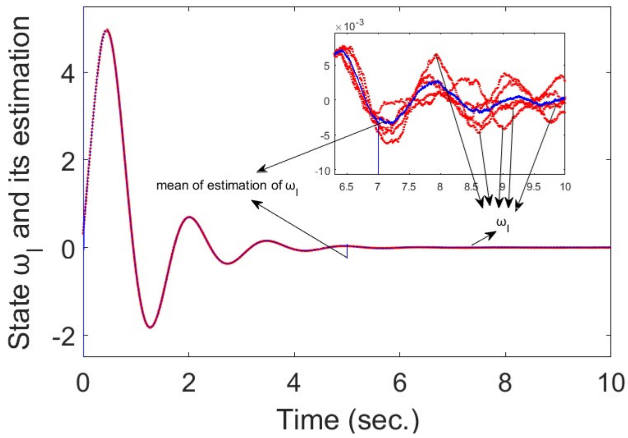

Fig. 5. w_l and the mean of its estimation.

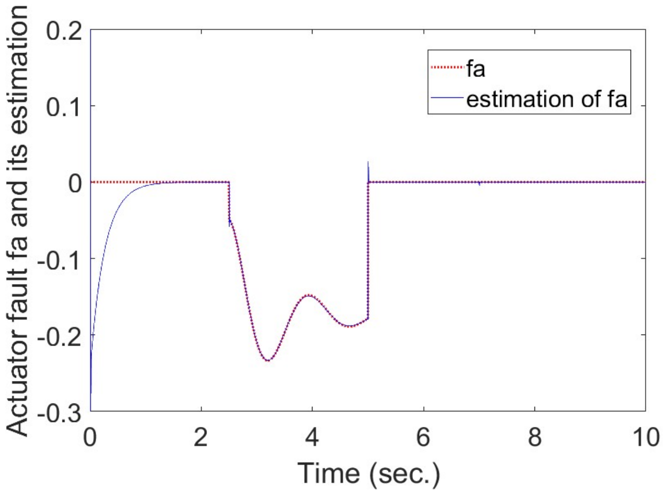

Fig. 6. f_a and its estimation.

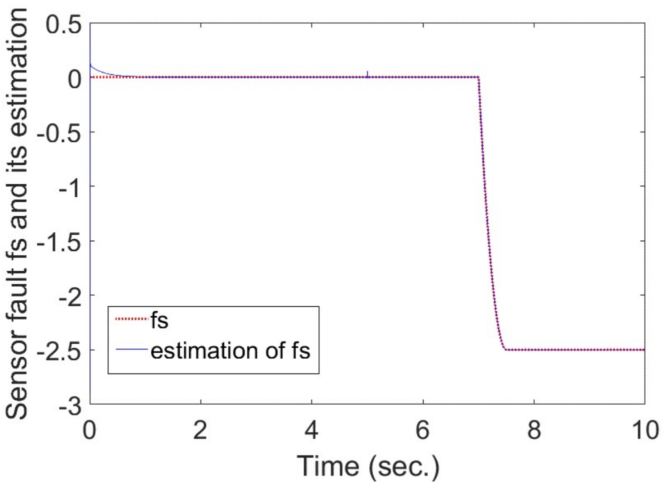

Fig. 7. f_s and its estimation.

As a result, the proposed fault estimation-based fault tolerant control techniques are effective.

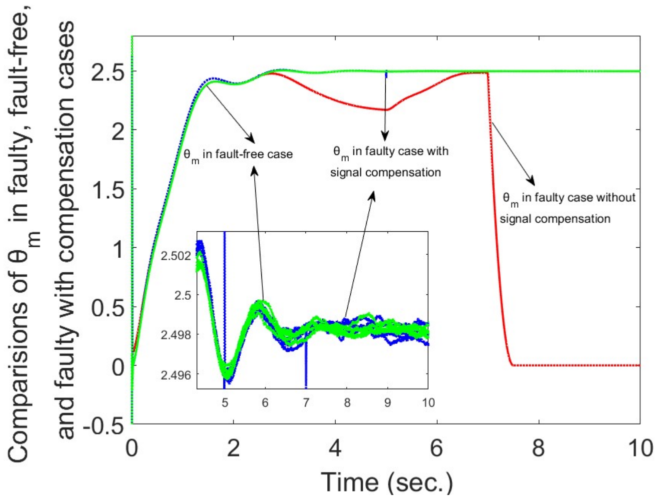

Fig. 8. Comparisons of the first output: fault-free output θ_m, output θ_m subjected to faults without tolerant control, and output subjected to faults after tolerant control.

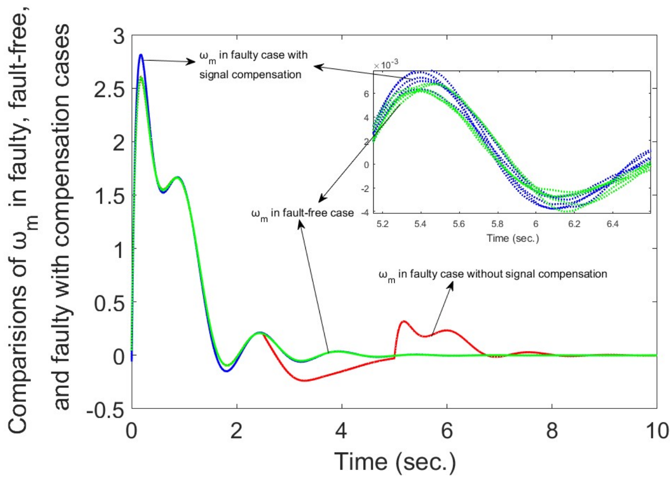

Fig. 9. Comparisons of the first output: fault-free output w_m, output w_m subjected to faults without tolerant control, and output subjected to faults after tolerant control.

6.2 *Example 2. The three-tank system*

In this section, the developed fault estimation and fault tolerant control methods are implemented to a laboratory setup DTS200 of the three-tank system with simultaneous unknown inputs, actuator fault, and stochastic parameter perturbations (see [7]). A T-S fuzzy model of the three-tank system can be built in the form of (2), where

$$A_1 = \begin{bmatrix} -0.0216 & 0 & 0.0216 \\ 0 & -0.0352 & 0.0216 \\ 0.0216 & 0.0216 & -0.0432 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.0153 & 0 & 0.0153 \\ 0 & -0.0289 & 0.0153 \\ 0.0153 & 0.0153 & -0.0305 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -0.0125 & 0 & 0.0125 \\ 0 & -0.0261 & 0.0125 \\ 0.0125 & 0.0125 & -0.0249 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0.00649 & 0 \\ 0 & 0.00649 \\ 0 & 0 \end{bmatrix},$$

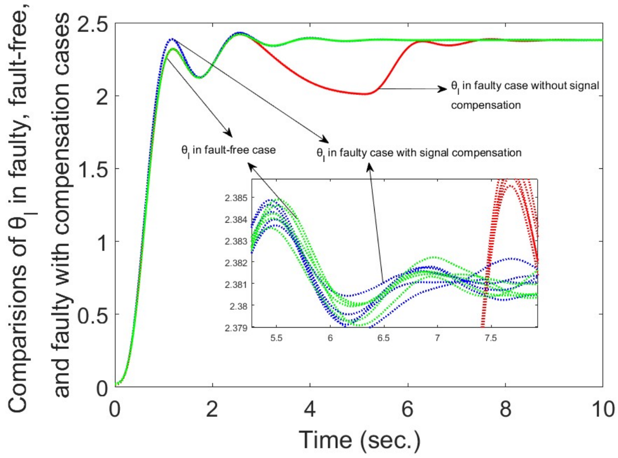

Fig. 10. Comparisons of the first output: fault-free output θ_l, output θ_l subjected to faults without tolerant control, and output subjected to faults after tolerant control.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_d = \begin{bmatrix} 0.1 \\ -0.2 \\ 0.5 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} -0.001 & 0 & -0.002 \\ 0 & -0.001 & 0.003 \\ -0.002 & 0.003 & 0.005 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} -0.002 & 0 & 0.001 \\ 0 & 0.002 & -0.004 \\ 0.001 & 0.005 & -0.003 \end{bmatrix},$$

$$W_3 = \begin{bmatrix} 0.003 & 0 & -0.001 \\ 0 & -0.001 & 0.002 \\ 0.002 & 0 & 0.001 \end{bmatrix}.$$

The system state $x = [x_1 \;\; x_2 \;\; x_3]^T$, where $x_i, i = 1,2,3$ represent the liquid level (cm) of the 3 tanks. The first two states are taken as the system outputs. The initial condition is $x_0 = [20 \;\; 10 \;\; 15]^T$ corrupted by random noises. The control inputs are flow rates of pump 1 and pump 2. The membership functions are

$$\begin{aligned} h_1(x_1) &= \frac{-x_1 + V_2}{2(V_2 - V_1)} \cdot sgn(-x_1 + V_2) + \frac{-x_1 + V_2}{2(V_2 - V_1)}, \\ h_2(x_1) &= \frac{-V_1 + x_1}{2(V_2 - V_1)} \cdot sgn(-x_1 + V_2) + \frac{-V_1 + x_1}{2(V_2 - V_1)} \\ &\quad + \frac{-x_1 + V_3}{2(V_3 - V_2)} \cdot sgn(-V_2 + x_1) + \frac{-x_1 + V_3}{2(V_3 - V_2)}, \\ h_3(x_1) &= \frac{-V_2 + x_1}{2(V_3 - V_2)} \cdot sgn(-V_2 + x_1) + \frac{-V_2 + x_1}{2(V_3 - V_2)}, \end{aligned}$$

where $V_1 = 15, V_2 = 20, V_3 = 25$.

The concerned actuator fault f_a is a deviation of value -10cm^3/s from 200 sec. to 400 sec., which is located on pump 2.Therefore $f = f_a, B_{f_1} = B_{f_2} = \begin{bmatrix} 0 \\ 0.00649 \\ 0 \end{bmatrix}$, and $D_f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Unknown input signal d is supposed to be random noises from -0.1 and 0.1. Observer gain H can be solved by (17) as

$$H = \begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \\ 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Choosing $\gamma = 1, \gamma_1 = 0.9, \sigma = 5$, and solving LMI problem (66), the controller and observer gains can be calculated as follows:

$$K_1 = K_2 = K_3 = 10^3 \times \begin{bmatrix} -2.2059 & -0.0080 & -0.0024 \\ -0.0080 & -2.2015 & -0.0024 \end{bmatrix},$$

$$L_{11} = 10^3 \times \begin{bmatrix} 0.0017 & 0.0012 \\ 0.0005 & 0.0022 \\ 0.0092 & 0.0052 \\ 0.3770 & 0.4228 \\ 0.6114 & 1.1225 \end{bmatrix},$$

$$L_{21} = 10^3 \times \begin{bmatrix} 0.0017 & 0.0012 \\ 0.0005 & 0.0023 \\ 0.0079 & 0.0021 \\ 0.3804 & 0.4260 \\ 0.6200 & 1.1418 \end{bmatrix},$$

$$L_{31} = 10^3 \times \begin{bmatrix} 0.0017 & 0.0012 \\ 0.0005 & 0.0023 \\ 0.0073 & 0.0008 \\ 0.3812 & 0.4281 \\ 0.6251 & 1.1497 \end{bmatrix},$$

and we can obtain $\theta = 10^{-9}$ such that $J_1^T T^T \bar{P} = N\bar{C}$ and $J_1^T T^T \bar{P} T \bar{W}_i = 0$ hold. Observer gains K_{f1}, K_{f2} and K_{f3} can be calculated according to (23). The sliding mode controller gain can be obtained as $L_{si} = TJ_1$.

Other observer gains T, L_{i2}, and R_i, where $i = 1, 2, 3$, can be calculated according to (13) to (15) respectively. Select $\rho_0 = 0.0001$, and $\varepsilon = 0.0001$, and using the Euler-Maruyama method ([12]) to simulate the standard Brownian motions (with 10 Brownian paths), we can obtain the simultaneous estimation results of full system states and concerned actuator fault as shown in Figs. 11-14. As the influences of actuator fault are mainly reflected by the second output y_2, comparisons of the output y_2 with and without signal compensations, and output in fault-free case are shown in Figs. 15-16 to illustrate fault tolerant control performance.

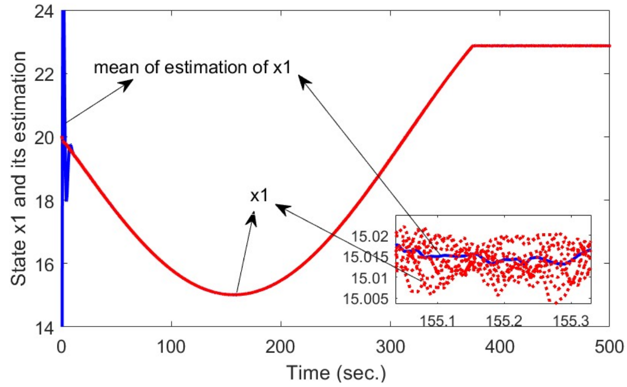

Fig. 11. x_1 and the mean of its estimation.

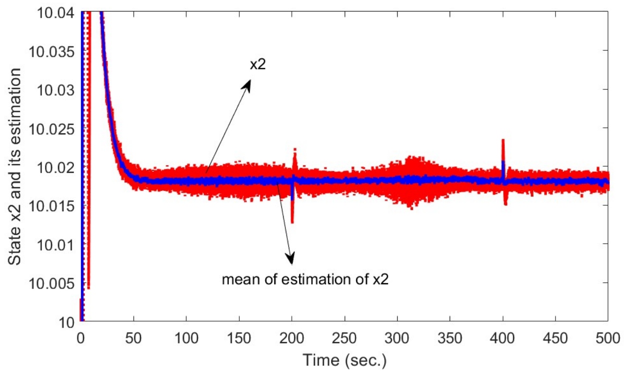

Fig. 12. x_2 and the mean of its estimation.

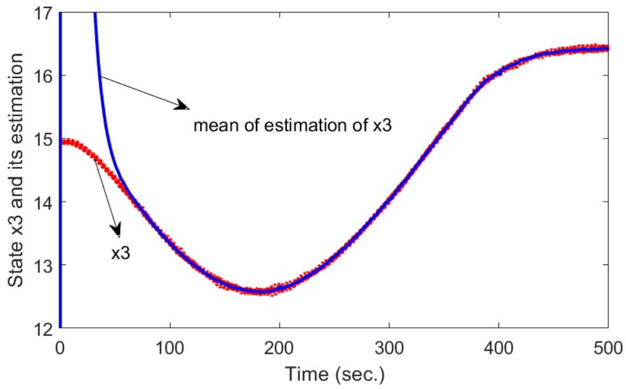

Fig. 13. x_3 and the mean of its estimation.

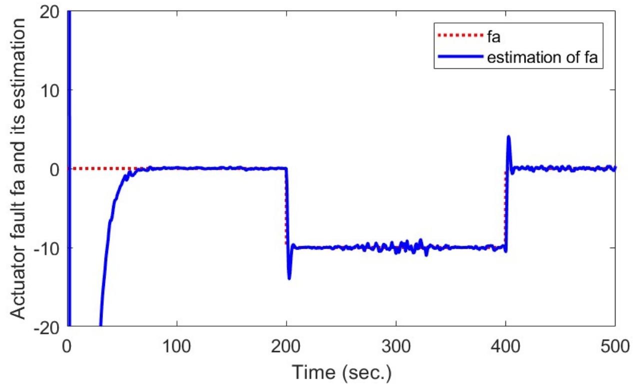

Fig. 14. f_a and its estimation.

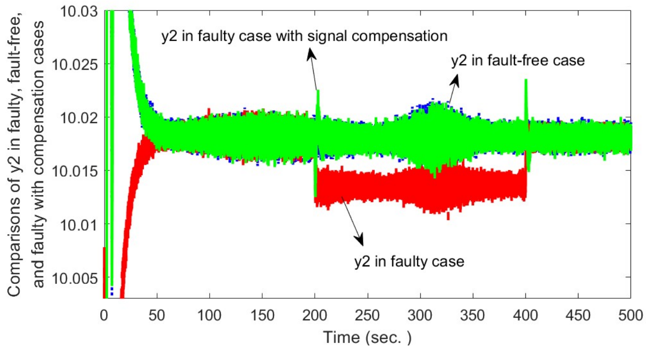

Fig. 15. Comparisons of y_2: fault-free output y_2, output y_2 subjected to faults without tolerant control, and output y_2 subjected to faults after tolerant control.

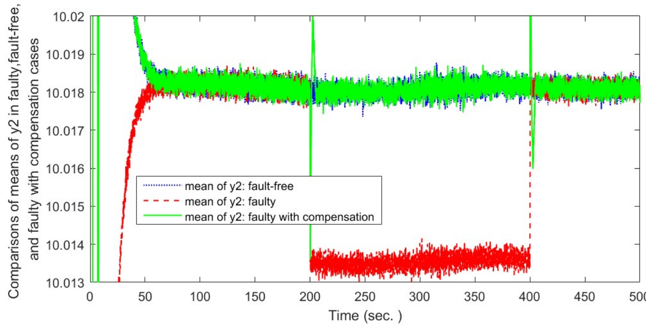

Fig. 16. Comparisons of means of y_2: fault-free output y_2, output y_2 subjected to faults without tolerant control, and output y_2 subjected to faults after tolerant control

From the above figures, we can find the estimations of both system states and actuator fault are well generated. When the system is subjected to actuator fault, the system cannot work normally. However, by implementing the developed fault tolerant control strategies, the deviation of output caused by faults can be mitigated satisfactorily, which has demonstrated that the designed integrated fault tolerant control techniques are effective.

7 Conclusion

In this work, a robust fault estimation and fault tolerant control technique is addressed for stochastic T-S fuzzy systems in presence of simultaneous actuator faults, sensor faults, unknown inputs and Brownian parameter perturbations. By integrating the augmented system approach, unknown input observer technique, sliding mode approach and linear matrix inequality optimization approach, the integrated fault estimation and tolerant control algorithms have been proposed. Simulation studies on a single-link manipulator and a three-tank model have well demonstrated the effectiveness of the proposed fault estimation and tolerant control technique. The proposed approach would have great potentials in applications to complex industrial processes with high nonlinearities and stochastic parameter perturbations.

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